Distorted distributions : construction methods and properties

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Distorted distributions: constructions and properties

1-Introduction

- The study of distortions is of general interest since they can be used for generating, in a flexible way, new families of copulas.
- In this section, we concentrate our attention on a transformation acting on bivariate copulas. Such a transformation has been considered several times in the literature,under different names like distortion or transformation of a copula by means of $g(.)$. It has originated from the study of distorted probability distribution functions (especially, power distortions), and has been considered by several authors like Frees and Valdez (1998); Durrleman et al. (2000); Genest and Rivest (2001); Durante and Sempi (2005a); Klement et al. (2005b,a); Morillas (2005); Charpentier (2008); Alvoni et al. (2009).

2-Distortion function

- A continuous increasing function $g: [0,\ 1] \to [0,1]$ such that $g(0) = 0\ and\ g(1) = 1$ is called *distortion* function.

- The dual transform $\tilde{g}(x) = 1 - g(1 - x)$ of a distortion function is called *dual distortion* function.
- In the case where g is convex, then we have what we call a *convex distortion function*. On the other hand, if g is concave, we have what we call *concave distortion function.*
- *Clearly, if* $g(.)$ is convex, then its dual $\tilde{g}(.)$ is concave.
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2.1-Some properties

- Let $g_1 \quad and \quad g_2$ be two distortion function, then for all $t \in [0, \quad 1]$:
- i) $g(t) = g_1(g_2(t))$ is a distortion function,
- ii) $g_1(t).\, g_2(t)$ is a distortion function,
- iii) $\alpha g_1(t) + (1 - \alpha) g_2(t)$ for all $0 < \alpha < 1$ is a distortion function,
- Iv) $\tilde{g}(t) = 1 - g(1 - t)$ is a distortion function (dual distortion function).
- Examples of distortion functions are summarized next page. Several other distortion functions can be found in Morillas (2005).
- Patricia M. Morillas. A method to obtain new copulas from a given one. *Metrika*, 61:169–184,2005.

2.2-Some examples of distortion functions

Distortion	$g(t)$	$g^{-1}(t)$	Convex constraints	Concave constraints
Proportional hazard	$t^{1/\alpha}$	t^{α}	$\alpha \geq 1$	$0 < \alpha \leq 1$
Exponential	$\frac{1-e^{-\alpha t}}{1-e^{-\alpha}}$	$\ln[1 - t(1 - e^{-\alpha})]$	$\alpha < 0$	$\alpha > 0$
Logarithm	$\frac{1}{\alpha}\ln[1-t(1-e^{\alpha})]$	$\frac{e^{\alpha t}-1}{e^{\alpha}-1}$	$\alpha < 0$	$\alpha > 0$
Wang transform	$\Phi[\Phi^{-1}(t) + \alpha]$	$\Phi[\Phi^{-1}(t) - \alpha]$	$\alpha \leq 0$	$\alpha \geq 0$
Dual-power	$1 - (1 - t)^{\alpha}$	$1 - (1 - t)^{1/\alpha}$	$0 < \alpha \leq 1$	$\alpha \geq 1$

2.3-Distorted distribution function

- Let X be a random variable with distribution function $F(x)$, the transform
- $F_{X^*}(x) = F_g(x) = g(F(x))$ defines a distribution function, which is called *distorted distribution* function and is the distribution function of X^*.
- In the actuarial literature, it is a more common practice to distort the survival function $\bar{F}(x) = 1 - F(x)$, instead of the distribution function.
- The dual distortion function defines a transformed distribution function
- $F_{\tilde{g}}(x) = \tilde{g}(F(x))$, which is called *dual distorted distribution* function.
- If we distort $F(x)$ with a distortion function $g(.)$, this implies that
- $\bar{F}_{X^*}(x) = 1 - g\left(1 - \bar{F}(x)\right)$.

3-Distortion premium principle

- In insurance pricing and in financial risk management, transformation of the distribution function typically represents a change in the probability measure. To illustrate in actuarial science, Wang (1996) defines a premium principle based on the concept of distortion function motivated by Yaari's dual theory of choice under risk; see Yaari (1987).
- The distortion premium principle associated with the distortion function g is then defined to be
- $R_g(X) = E(X^*),$ (1)
- the expectation under the distorted probability measure i.e.
- The random variable X^* having to distribution function $F_g(x)$.

3.1-Distortion premium principle

- Note that for insurance premium purposes, this distorted expectation must be at least equal to the expectation under the original probability measure
- $$R_g(X) \geq E(X).$$
- Such is the case only when $g(x)$ is convex. To see this, if $g(x)$ is indeed convex, then direct application of Jensen's inequality leads us to:
- $F_g(x) = g\big(F(x)\big) = g\big(EI_{[X\leq x]}\big) \leq Eg\big(I_{[X\leq x]}\big)$
- $= g(0).P[X > x] + g(1).P[X \leq x] = F(x).$
- So, $R_g(X) = \int_0^\infty \bar{F}_g(x)dx \geq \int_0^\infty \bar{F}(x)dx = E(X).$
- Thus, clearly, $R_g(X) - E(X) \geq 0,$ (2)
- and this difference is often referred to as the risk premium.

4-Construction of distortion functions

- Let N be a nonnegative discrete random variable with probability moment function $g_N(t) = E(t^N),\ 0 < t < 1,$ then $g_N(t)$ is a distortion function.
- For baseline distribution function $F(x)$, the function $F_g(x) = g\big(F(x)\big)$ is a distorted distribution function.
- Example: Let $P(N = k) = \theta(1-\theta)^{k-1},\ k = 1,2,3,\ldots\ldots,$ then
- $g(t) = \frac{\theta t}{1-(1-\theta)t},\ 0 \le \theta \le 1,\ \ 0 \le t \le 1$
- So, the function $F_g(x) = \frac{\theta F(x)}{1-(1-\theta)F(x)},\qquad 0 \le \theta \le 1,$ is a distorted distribution.

4.1-probability generating function Method

- Let $P(N=k)=\theta(1-\theta)^{k-1},\ k=1,2,3,\ldots..,$ and $\{X_n, n\geq 1\}$ be a iid sequence of random variables with df $F(x)$, also suppose that N is independent of X_n.
- If $U_N = min\{X_1, \ldots..X_N\}$ and $V_N = max\{X_1, \ldots..X_N\}$ then
- $F_{V_N}(x) = \frac{\theta F(x)}{1-(1-\theta)F(x)}=g(F(x))$ and $\bar{F}_{V_N}(x) = \frac{\theta \bar{F}(x)}{1-(1-\theta)\bar{F}(x)}=g(\bar{F}(x))$.
- Where $g(t) = \frac{\theta t}{1-(1-\theta)t}$.
- Marshall, A.N. and Olkin, I.(1997) A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. Biometrica 84, 611-652.
- Krishna,E., Jose, K.K. and M.M.Ristic (2013) Applications of Marshall-Olkin Frechet distribution. Comm. Stat. simulation and Computation, 42: 76-89.

4.1. probability generating function Method

- Let $P(N=k) = -\frac{\theta^k}{n.\ln(1-\theta)}, 0 < \theta < 1,\ k = 1,2,3, \ldots\ldots,$ and $\{X_n, n \geq 1\}$ be a iid sequence of random variables with df $F(x)$, also suppose that N is independent of X_n.
- If $U_N = min\{X_1, \ldots\ldots X_N\}$ and $V_N = max\{X_1, \ldots\ldots X_N\}$ then $g(t) = \frac{\ln(1-\theta t)}{\ln(1-\theta)},\ 0 \leq t \leq 1$
- $F_{V_N}(x)$=$g(F(x)) = \frac{\ln(1-\theta F(x))}{\ln(1-\theta)}$ and $\bar{F}_{V_N}(x)$=$g(\bar{F}(x)) = \frac{\ln(1-\theta \bar{F}(x))}{\ln(1-\theta)}$.

4.2-Mittag – Leffler function

- Let N be a nonnegative discrete random variable with probability function
- $P(N=k)=\frac{1}{(E_\alpha(1)-1)\Gamma(k\alpha+1)}$, $k=1,2,\ldots\ldots,$ where $E_\alpha(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+1)}$, α *is complex number with* $Re(\alpha)>0$, is Mittag-Leffler function. Then
- $g(t)=\frac{1}{E_\alpha(1)-1}\sum_{n=1}^{\infty}\frac{t^n}{\Gamma(\alpha n+1)}=\frac{E_\alpha(t)-1}{E_\alpha(1)-1},\ 0\le t\le 1.$
- If $\{X_n,n\ge 1\}$ *is a sequence of iid random variables with df* $F(x)$, and N is independent of X_n, then
- $F_U(x)=g\big(F(x)\big)=\frac{E_\alpha(F(x))-1}{E_\alpha(1)-1}$ and $\bar{F}_V(x)=g\big(\bar{F}(x)\big)=\frac{E_\alpha(\bar{F}(x))-1}{E_\alpha(1)-1}$
- Where $U_N=min\{X_1,\ldots\ldots X_N\}$ and $V_N=max\{X_1,\ldots\ldots X_N\}$
- It is easy to show that If $\alpha=1$ then N is a random variable with truncated Poisson distribution.

4.3-Stieltjes Laplace transform method

- Let X be a nonnegative random variable with distribution function $F(x)$, and
- $\psi(t) = E(e^{-tX}) = \int_0^\infty e^{-tx} dF(x), \quad t \in [0, \infty]$ be Stieltjes Laplace transform of $F(x)$.
- Then $g(t) = \exp[-\psi^{-1}(t)], \; t \in [0, 1]$ is a increasing convex distortion function.
- Example. Suppose that $\bar{F}(x) = \exp[-\theta x]$, then $\psi(t) = \frac{\theta}{\theta+t}$ *and* $\psi^{-1}(t) = \theta(\frac{1}{t} - 1)$, so
$$g(t) = \exp\left[-\theta\left(\frac{1}{t} - 1\right)\right],$$
- therefore, for base line distribution $F(x)$, $F_g(x) = \exp\left[-\theta\frac{\bar{F}(x)}{F(x)}\right]$ is a distorted distribution function.

4.4- Sum of a random number of random variables

- Let $\{X_n, n \geq 1\}$ be a independent identically distributed sequences with partial sums $S_n = \sum_{k=1}^{n} X_k,\ \ n \geq 1$ and N be a non-negative integer valued random variable independent of sequence $\{X_n, n \geq 1\}$, then $g_{S_N}(t) = g_N\left(g_{X_1}(t)\right)$.
- Example : Let N is a random variable with truncated Poisson distribution with parameter θ and X_1 is a Geometric random variable with parameter p. Then
- $$g_{S_N}(t) = \frac{exp\left(\frac{-\theta(1-t)}{1-qt}\right) - \exp(-\theta)}{1-\exp(-\theta)}$$
- Hence
  - $$F_g(t) = \frac{exp\left(\frac{-\theta(1-F(t))}{1-qF(t)}\right) - \exp(-\theta)}{1-\exp(-\theta)} \text{ and } \bar{F}_g(t) = \frac{exp\left(\frac{-\theta F(t)}{1-q\bar{F}(t)}\right) - \exp(-\theta)}{1-\exp(-\theta)}$$

4.5-Combine two distortion functions

- It is well known that If g_1 $\ and\ $ g_2 be two distortion function, then
- $g(t) = g_2(g_1(t))$ is a distortion function.
- Example. Let $g_1(t) = \frac{\theta_1 t}{1-(1-\theta_1)t}$, $0 < \theta_1 < 1$ $\quad and \quad$ $g_2(t) = t\left(1 + \theta_2(1-t)\right)$,
- $-1 < \theta_2 < 1$, then $g(t) = g_2\left(g_1(t)\right) = \frac{\theta_1 t}{1-(1-\theta_1)t}\left(1 + \theta_2\left(1 - \frac{\theta_1 t}{1-(1-\theta_1)t}\right)\right)$, is a distortion function, so
- $$F_g(x) = g\left(F(x)\right) = \frac{\theta_1 F(x)}{1-(1-\theta_1)F(x)}\left(1 + \theta_2\left(1 - \frac{\theta_1 F(x)}{1-(1-\theta_1)F(x)}\right)\right)$$
- Is a distorted distribution function.

5-Some distorted distributions

- Let $f(t)$ be the density function of a random variable $T \in [a,\ b]$
- for $-\infty < a < b < \infty$ and let W [G(x)] be a function of the distribution function of a random variable X such that W [G(x)] satisfies the following
- conditions:
- $$\begin{cases} i)\quad W\big(G(x)\big) \in [a, \quad b], \\ ii)\, W\big(G(x)\big) is\ differentiable\ and\ non-dicresing, \\ iii)\ W\big(G(x)\big) \to a\ as\ \ x \to -\infty\ \ and\ W\big(G(x)\big) \to b\ as\ \ x \to \infty \end{cases}$$
- Then $g(t) = \int_a^{W(t)} f(x)dx, \quad t \in [0, \quad 1]$ is a distortion function.

5-Some distorted distribution

- Example : Let $W(t) = t^{\alpha}$ and $f(t) = 1 + \lambda - 2\lambda t, \quad t \in (0,\ 1)\ and\ |\lambda| \leq 1$, then
- $$g(t) = (1+\lambda)t^{\alpha} - \lambda t^{2\alpha}, \quad \alpha > 0.$$
- So if $F(x)$ is a baseline distribution, then
- $$F_g(x) = 1 - [1 - \lambda F^{\alpha}(x)][1 - F^{\alpha}(x)]$$
- is a distorted distribution function.

5-1. Transmuted Dagum Distribution

- The function $F_g(x) = F(x)[1 + \theta\bar{F}(x)], \quad -1 \leq \theta \leq 1$, is a distorted distribution with corresponding distortion function as: $g(t) = t[1 + \theta(1 - t)], \quad 0 \leq t \leq 1.$
- Example : Let $F(x) = \left(1 + \left(\frac{x}{\beta}\right)^{-\alpha}\right)^{-\rho}, x \geq 0; \ \alpha, \beta, \rho > 0,,$
- Then $F_g(x) = F(x)\left[1 + \theta\left(1 - F(x)\right)\right]$, is called Transmuted Dagum Distribution.
- Mirza Naveed Shahzad and Zahid Asghar
- Transmuted Dagum distribution: A more flexible and board shaped hazard function model. Hacettepe Journal of Mathematics and Statistics, Vol 45(1) (2016) 227-244

5.2- The beta generalized gamma distribution

- The function $F_g(x) = \int_0^{F(x)} \frac{1}{Bet(a,b)} x^{a-1} \ (1-x)^{b-1} dx$, is a distorted distribution with corresponding distortion function as:
- $g(t) = Bet(a,b,t) = \int_0^{t} \frac{1}{Bet(a,b)} x^{a-1} \ (1-x)^{b-1} dx, \quad a < 1, \quad b > 1.$
- The density function corresponding to $F_g(x)$ is:
- $f_g(x) = \frac{f(x)}{Bet(a,b)} [F(x)]^{a-1} [1-F(x)]^{b-1}, \ a,b>0.$
- **Gauss M. et al.**(2012)**.The beta generalized gamma distribution. Statistics: 1-13**

6-Distorted bivariate distributions

- Let $g(t)$ be a continuous increasing convex distortion function and $F(x,y)$ be
- A baseline distribution, then the distorted bivariate function
- $$F_g(x,y) = g\big(F(x,y)\big),$$
- Is a bivariate distribution that called distorted bivariate distribution with margins
$$F_{1g}(x) = g\big(F_1(x)\big), F_{2g}(y) = g\big(F_2(y)\big).$$
- Example: The function $g(t) = \exp[\theta(t-1)]$, $0 < t < 1$, is a continuous increasing convex distortion function, So if $F(x,y)$ is a bivariate distribution then
- $F_g(x,y) = \exp[\theta(F(x,y) -)]$, is distorted bivariate distribution function.

7-Distorted copula functions

- Let $g(t)$ be increasing convex distortion function and $F(x, y)$ be baseline distribution function, then applying the Sklar's Theorem, it is easy to show that:
- $C_g(u, v) = g(C(g^{-1}(u), g^{-1}(v))$.
- This copula is called distorted copula function.
- Klement at al.(2004) The bivariate function $C_g(u, v) = g(C(g^{-1}(u), g^{-1}(v))$
- Is a copula if and only if: $g: [0,\ 1] \rightarrow\ [0,\ 1],\ \ with\ g(0) = 0,\ g(1) = 1\ \ is$
- Convex.

7.1. Example

- 1)-Let $g(t) = \frac{\theta t}{1-(1-\theta)t}$, $0 < \theta < 1$, then $g^{-1}(t) = \frac{t}{\theta+(1-\theta)t}$, so if
- $C(u,v) = \Pi(u,v)$ then by simple calculation we can write
- $C_g(u,v) = g\left(g^{-1}(u).\,g^{-1}(v)\right) = \frac{\theta g^{-1}(u)g^{-1}(v)}{1-(1-\theta)g^{-1}(u)g^{-1}(v)} = \frac{uv}{1-(1-\theta)(1-u)(1-v)}$,
- this copula is AMH(θ).
- Exercise: Let $g(t) = \frac{1-e^{-\alpha t}}{1-e^{-\alpha}}$, $\alpha > 0$, $0 \le t \le 1$ and $C(u,v) = \Pi(u,v)$, Then find distortion copula generated by $g(t)$ $\quad and \quad base\ copula \quad C(u,v) = \Pi(u,v)$.

7.1-A new distortion function

- Let $C(u, v)$ be a copula function, then for every $\alpha \in [0, \;\; 1]$ the function
- $g_\alpha(t) = \frac{C(\alpha,\ t)}{\alpha}, \quad for\ all\ \ t \in [0, \;\; 1]$ is a distortion. Since,
- $g_\alpha(0) = 0, \;\; g_\alpha(1) = 1, \quad and \quad g'_\alpha(t) = \frac{\partial C(\alpha, t)}{\alpha . \partial t} \geq 0.$
- Example 1: Let $C(u, v) = uv[1 + \theta(1-u)(1-v)] \quad |\theta| \leq 1.$ Then
- $g_\alpha(t) = t[1 + \lambda(1-t)], \quad \lambda = \theta(1-\alpha).$
- Example 2: Let $C(u, v) = uv.exp[\theta(1-u)(1-v)] \quad |\theta| \leq 1.$ Then
- $g_\alpha(t) = t.exp[\lambda(1-t)], \quad \lambda = \theta(1-\alpha).$
- Example 3: Let $C(u, v) = \frac{uv}{[1-\theta(1-u)(1-v)]} \quad |\theta| \leq 1.$ Then
- $g_\alpha(t) = \frac{t}{1+\lambda(1-t)}, \quad \lambda = \theta(1-\alpha).$

7.2-GMC copula

- Let $g(t) = \mathrm{t.}\exp[\theta(t-1)]$, $0 \leq \theta \leq 1$, $0 \leq t \leq 1$, then
- $C^{+}(u,v) = uv\frac{g(uv)}{g(u)g(v)} = uvexp[\theta(1-u)(1-v)]$, is a GMC copula with positive dependence structure,
- $C^{-}(u,v) = uv\frac{g(u)g(v)}{g(uv)} = uvexp[-\theta(1-u)(1-v)]$, is a GMC copula with negative dependence structure,
- Therefore for all $\theta \in [-1, \ 1]$,
- $C_{\theta}(u,v) = (uvexp[(1-u)(1-v))^{\frac{1+\theta}{2}}(uv\exp[-(1-u)(1-v))^{\frac{1-\theta}{2}}$
- $\qquad = uv\exp[\theta(1-u)(1-v)].$
- Is called GMC copula.

8-Generalized distorted distributions

- The generalized distorted distribution associated to n distribution functions
- $F_1, F_2, \dots, F_n$ and to an increasing continuous multivariate function
- $Q: [0,\ 1]^n \rightarrow [0,\ 1]$ such that $Q(0,0,\dots,0) = 0\ \ and\ \ Q(1,1,\dots,1) = 1$
- is defined by $F_Q(t) = Q(F_1(t), F_2(t), \dots, F_n(t))$. (1)
- The function $F_Q(t)$ defined above is always a proper (univariate) distribution function (it is increasing, right-continuous and satisfies
- $F_Q(-\infty) = Q(0,0,\dots,0) = 0$ and $F_Q(\infty) = Q(1,1,\dots,1) = 1$.

8-Generalized distorted distributions

- Moreover, if Q is strictly increasing in each variable and $F_1, F_2, \ldots, F_n$ have the same support S, then $F_Q(t)$ also has the same support S.
- We have a similar expression for the respective reliability functions (RF)
- $\bar{F}_Q(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t)),$ (2)
- where $\bar{F}_i(t) = 1 - F_i(t)$ *and* $\bar{F}_Q(t) = 1 - F_Q(t)$.
- Where $\bar{Q}(u_1, u_2, \ldots, u_n) = 1 - Q(1 - u_1, 1 - u_2, \ldots, 1 - u_n)$ Is the multivariate dual distortion function.
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8-Generalized distorted distributions

- The function $\bar{Q}$ is also a multivariate distortion function, that is, it satisfies the same properties as Q. Hence the function $\bar{F}_Q$ defined above is always a proper (univariate) reliability function. These two representations are equivalent but, sometimes, it is better to use Eq. (2) instead of Eq. (1).
- Of course, if n = 1 (or if $F_1 = F_2 = \cdots F_n$), then we obtain the distorted distribution
- defined in Eqs. (1) and (2). Note that representations in Eqs. (1) and (2) are similar to copula representations for multivariate distributions.
- Actually the copulas are valid distortion functions but the distortion functions are not necessarily copulas.

9-Application in coherent systems

- Navarro, J. and Del,Aguila, Y.(2017) Stochastic comparison of distorted distributions, coherent systems and mixtures with ordered components. Metrika, 80: 627-648.

- Navarro, J. et al.(2016) Preservation of stochastic orders under the formation of generalized distorted distributions. Applications in coherent systems. Meth. Comp.Appl.Prob. 18: 529-545.

9-An application in coherent systems

- Let $X_1, X_2, \dots, X_n$ be component lifetimes of a coherent system with lifetime T
- Then the system distribution $F_T(t)$ can be written as
- $F_T(t) = Q(F_1(t), F_2(t), \dots, F_n(t)),$
- For all $t,$ that is a generalized distorted distribution of the component distribution functions, where the distortion function $Q(.,.,\dots,.)$ only depends on the structure of the system and on the copula of the random vector $(X_1, X_2, \dots, X_n)$. Also, reliability function of the structure is:
- $\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \dots, \bar{F}_n(t)),$
-

9.1-Some examples

- Example 1:-Let X_1, X_2, X_3 be independent component lifetimes of a coherent system with lifetime $T = \max\{X_1, \min\{X_2,\ X_3\}\}$, then
- $\bar{Q}(u_1, u_2, u_3) = u_1 + u_2u_3 - u_1u_2u_3.$
- Example 2: Let X_1, X_2, X_3 be dependent component lifetimes of a coherent system with corresponding copula $C(u_1, u_2, u_3)$ and lifetime
- $T = \max\{X_1, \min\{X_2,\ X_3\}\}$, then
- $\bar{Q}(u_1, u_2, u_3) = C(u_1, 1,1) + C(1, u_2, u_3) - C(u_1, u_2, u_3).$
- In particular case if
- $C(u_1, u_2, u_3) = u_1u_2u_3[1 + \theta(1 - u_1)(1 - u_2)(1 - u_3)], \quad |\theta| \leq 1$
- Then
- $\bar{Q}(u_1, u_2, u_3) = u_1 + u_2u_3 - u_1u_2u_3[1 + \theta(1 - u_1)(1 - u_2)(1 - u_3)].$