Chapter 1

Definitions and Basic Properties

Contents

It is well known and easily verified that $F_1(X)$ and $F_2(Y)$, where F_1 and F_2 are the marginals distributions of X and Y respectively, are two uniform variables if F_1 and F_2 are continuous. Hence if the marginals F_1 and F_2 of the bivariate distribution F are continuous, there exists a unique copula, which is a cumulative distribution function, with its marginals being uniform. Formally, a function $C : [0,1]^2 \to [0,1]$ such that

$$F(x, y) = C(F_1(x), F_2(y)) \tag{1.1}$$

is a copula. On other hand, if $C(u_1, u_2)$ and continuous functions F_1 and F_2 are given, then there exists a bivariate distribution function F such that

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)), \tag{1.2}$$

where, $F_i(t)$, $i = 1, 2$ is continuous and non decreasing, but could be constant on some intervals. In that case, one defines a quasi-inverse by

$$F_i^{-1}(t) = \inf\{x : F_i(x) \geq t\}. \tag{1.3}$$

Lemma 1.0.1. *Let H be a joint distribution function with marginal F and G. Then for all $(x_i, y_i) \in R^2, i = 1, 2$:*

$$|H(x_2, y_2) - H(x_1, y_1)| \leq |F(x_2) - F(x_1)| + |G(y_2) - G(y_1)|.$$

Proof. See Chapter 2 Nelsen (2006).

Using copulas allows us to separate the study of dependence from the study of the marginals, since one is then reduced to study of the relation between two uniform variables. The purpose of this section is to present the results on copulas scattered in diverse literature with the emphasis on dependence concepts and properties.

Definition 1.0.2. A bivariate copulas is a function $C : [0,1]^2 \to [0,1]$ subject to

i) $C(x, 0) = C(0, y) = 0$, for all $x, y \in [0, 1]$.

ii) $C(x,1) = x, \quad C(1,y) = y$, for all $x, y \in [0,1]$.

iii) C is joint-increasing (i.e. for every 2-box $J = [x_1, x_2] \times [y_1, y_2] \in [0,1]^2$, the associated C-volume $V_C(J)$ satisfies $V_C(J) = C(x_2, y_2) + C(x_1, y_1) - C(x_1, y_2) - C(x_2, y_1) \geq 0$).

1.0.1 Some properties

In this section, we discuss some elementary properties of copula function.

Theorem 1.0.3. *Let C be a copula. Then for every $(u,v) \in I^2$,*

$$\max\{u+v-1, 0\} \leq C(u,v) \leq \min\{u,v\}, \qquad \forall u, v \in I.$$

Proof. By properties of copula $C(u,v)$ we can write $\forall u, v \in I^2$

$$0 \leq C(u,v) \leq C(u,1) = u, \qquad 0 \leq C(u,v) \leq C(1,v) = v$$

so, $0 \leq C(u,v) \leq \min\{u,v\}$. Also, $0 \leq V_C([u,1] \times [v,1]) = 1 - u - v$.
These imply that:

$$\max\{u+v-1, 0\} \leq C(u,v) \leq \min\{u,v\}, \qquad \forall u, v \in I.$$

□

Inequality $\max\{u+v-1, 0\} \leq C(u,v) \leq \min\{u,v\}$ is the copula version of the Fréchet-Hoeffding bounds inequality, which we shall encounter later in terms of distribution functions. In the litretures refer to $M(u,v) = \min\{u,v\}$ as the Fréchet-Hoeffding upper bound and $W(u,v) = \max\{u+v-1, 0\}$ as the Fréchet-Hoeffding lower bound. A third important copula that we will frequently encounter is the product copula $\Pi(u,v) = uv$.
The following theorem, which follows directly from Lemma 1, establishes the continuity of copulas via a Lipschitz condition on I^2.

Theorem 1.0.4. *Let C be a copula. Then for all $(u_i, v_i) \in I^2, \quad i = 1, 2$*

$$|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|.$$

Hence C is uniformly continuous on its domain

The sections of a copula will be employed in the construction of copulas in the next chapter, and will be used in Chapter 5 to provide interpretations of certain dependence properties:

Definition 1.0.5. Let $C(u, v)$ be a copula, and let a be any number in I. The horizontal section of C at a is the function from I to I given by $h(t) = C(t, a)$; the vertical section of C at a is the function from I to I given by $v(t) = C(a, t)$; and the diagonal section of C is the function δ_C from I to I defined by $\delta_C(t) = C(t, t)$. The following corollary is an immediate consequence of Lemma 1 and Differentiability copula.

Corollary 1.0.6. *The horizontal, vertical, and diagonal sections of a copula C are all nondecreasing and uniformly continuous on I.*

We conclude this section with the two theorems concerning the partial derivatives of copulas. The word "almost" is used in the sense ofLebesgue measure.

Theorem 1.0.7. *Let C be a copula. Then*

- *i) For any $v \in I$, the partial derivative $\frac{\partial C(u,v)}{\partial u}$ exists for almost all u, and for such v and u,*

$$0 \leq \frac{\partial C(u, v)}{\partial u} \leq 1,$$

- *ii) Similarly, for any $u \in I$, the partial derivative $\frac{\partial C(u,v)}{\partial v}$ exists for almost all v, and for such u and v,*

$$0 \leq \frac{\partial C(u, v)}{\partial v} \leq 1,$$

- *iii) Furthermore, the functions $p(v) = \frac{\partial C(u,v)}{\partial u}$ and $p(u) = \frac{\partial C(u,v)}{\partial v}$ are defined and nondecreasing almost everywhere on I. Let C be a copula.*

- *iv) If $\frac{\partial C(u,v)}{\partial v}$ and $\frac{\partial^2 C(u,v)}{\partial u \partial v}$ are continous on I^2 and $\frac{\partial C(u,v)}{\partial u}$ exists for all $u \in (0, 1)$ when $v = 0$, then $\frac{\partial C(u,v)}{\partial u}$ and $\frac{\partial^2 C(u,v)}{\partial u \partial v}$ exist in I^2 and*

$$\frac{\partial^2 C(u, v)}{\partial u \partial v} = \frac{\partial^2 C(u, v)}{\partial v \partial u}$$

Proof. Nelsen(2006), Section 2.2, Theorems 2.2.7 and 2.2.8. □

Corollary 1.0.8. *Under the assumtions of the Theorem, the density function of copula* $C(u,v)$ *given by:* $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$

1.0.2 Sklar Theorem

he theorem in the title of this section is central to the theory of copulas and is the foundation of many, if not most, of the applications of that theory to statistics. Sklar's theorem elucidates the role that copulas play in the relationship between multivariate distribution functions and their univariate margins. Thus we begin this section with a short discussion of distribution functions.

Theorem 1.0.9. *(Sklar, 1959) Let* F *be a joint distribution function with marginal* F_1 *and* F_2*. Then, there exists a copula* C *subject to*

$$F(x,y) = C(F_1(x), F_2(y)); \forall x, y \in R \tag{1.4}$$

$i)$ If F_1 and F_2 are continuous, then C is unique. Otherwise, C is uniquely determined on Ran $F_1 \times$ Ran F_2.
$ii)$ Conversely, if C is a copula and F_1 and F_2 are distribution functions, then the function $F(x,y) = C(F_1(x), F_2(y))$ is a joint distribution with marginal F_1 and F_2.

Proof. By appliying lemma 1 and since the joint distribution $F(x,y)$ is 2-increasing, having to margins F_1 anf F_2 and increasing in arguments the proof completes.

□

Corollary 1.0.10. *If* $F(x,y)$ *is a continuous bivariate distribution function with marginal* F_1 *and* F_2 *and quantile functions* F_1^{-1} *and* F_2^{-1} *then for all* $u, v \in [0,1]$, $C(u,v) = F(F_1^{-1}(u), F_2^{-1}(v))$ *is the unique choice. Where* $F^{-1}(u) = \sup\{x : \ F(x) \leq u\} = \inf\{x : \ F(x) \geq u\}$.

Example 1.0.11. We consider the following two cases.

i) FGM family: Let (X, Y) be a random vector with distribution function

$$F(x, y) = F_1(x)F_2(y)[1 + \theta\bar{F}_1(x)\bar{F}_2(y)]$$

then, $C(u, v) = uv[1 + \theta(1 - u)(1 - v)]$.

ii) Gumbel family: Let (X, Y) be a random vector with distribution function

$$\bar{F}(x, y) = \exp\{-(x + y + \theta xy)\},\ x \geq 0,\ y \geq 0,\ 0 \leq \theta \leq 1.$$

Then, we can show that

$$C(u, v) = 1 - u - v + (1 - u)(1 - v)\exp\{-\theta \ln(1 - u)\ln(1 - v)\}.$$

iii) Gaussian Copula: Let

$$f(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}\sigma_1\sigma_2} \exp\left[-\frac{1}{2(1 - \rho^2)}\left[\frac{(x - \mu_1)^2}{\sigma_1^2} + \frac{(y - \mu_2)^2}{\sigma_2^2} - 2\rho\frac{(x - \mu_1)(y - \mu_2)}{\sigma_1\sigma_2}\right]\right],$$

then

$$C_\rho(u, v) = N_\rho\left(\Phi_1^{-1}(u), \Phi_2^{-1}(v)\right),$$

where $N_\rho(t, s)$ is standard bivariate Normal distribution function.

1.1 Copula and random variables

In this section, we will use the term “random variable” in the statistical rather than the probabilistic sense; that is, a random variable is a quantity whose values are described by a (known or unknown) probability distribution function. Of course, all of the results to follow remain valid when a random variable is defined in terms of measure theory, i.e., as a measurable function on a given probability space.

Theorem 1.1.1. *Let X and Y be random variables with distribution functions F and G, respectively, and joint distribution function H. Then there exists a copula C such that $H(x, y) = C(F(x), G(y)); \forall x, y \in R$ holds.*

i) If F and G are continuous, C is unique. Otherwise, C is uniquely determined on Ran $F \times$ Ran G. The copula C in this Theorem will be called the copula of X and Y, and denoted by $C_{X,Y}$ when its identification with the random variables X and Y is advantageous.
ii) Conversely, if C is a copula and F and G are distribution functions, then the function $H(x, y) = C(F(x), G(y))$ is a joint distribution function with marginal F and G.

Proof. If F anf G are continous then $U = F(X)$ and $V = G(Y)$ are standard uniform random variables, now if the random vector (U, V) having to joint distribution $C(u, v)$ then

$$H(x, y) = P[X \leq x, Y \leq y] = P[F(X) \leq F(x), G(Y) \leq G(y)] = C(F(x), G(y)),$$

where $C(u, v)$ is unique. □

The following theorem shows that the product copula $\prod(u, v) = uv$ characterizes independent random variables when the distribution functions are continuous.

Theorem 1.1.2. *Let X and Y be continuous random variables. Then X and Y are independent if and only if $C_{X,Y}(u, v) = \prod(u, v)$.*

Proof. The proof follows from above and the observation that X and Y are independent if and only if $H(x, y) = F(x)G(y), forallx, y \in R^2$. □

Corollary 1.1.3. *Under the assumptions of Theorem 1.17, we obtain*

i) $f(x, y) = f_1(x)f_2(y)c(F_1(x), F_2(y))$.

ii) $c(u, v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$.

Much of the usefulness of copulas in the study of nonparametric statistics derives from the fact that for strictly monotone transformations of the random variables, copulas are either invariant or change in predictable ways. Recall that if the distribution function of a random variable X is continuous, and if $\alpha(x)$ is a strictly monotone function whose domain contains $RanX$, then the distribution function of the random variable $\alpha(X)$ is also continuous. We treat the case of strictly increasing transformations first.

Theorem 1.1.4. *(Nelsen, 2006) Let X and Y be two continuous random variables with copula function C_{XY}. If $\alpha(.)$ and $\beta(.)$ are strictly increasing on RanX and RanY respectively, $C_{\alpha(X),\beta(Y)} = C_{X,Y}$. Thus, $C_{X,Y}$ is invariant under strictly increasing transformation of X and Y.*

Proof. Let $\alpha(X) \sim G_1$ and $\beta(Y) \sim G_2$. Then $G_1(x) = F_1(\alpha^{-1}(x))$ and $G_2(y) = F_2(\beta^{-1}(y))$. Since $\alpha(.)$ and $\beta(.)$ are strictly increasing, we have

$$\begin{aligned}
C_{\alpha(X),\beta(Y)}(G_1(x), G_2(y)) &= P[\alpha(X) \leq x, \beta(Y) \leq y] \\
&= P[X \leq \alpha^{-1}(x), Y \leq \beta^{-1}(y)] \\
&= F(\alpha^{-1}(x), \beta^{-1}(y)) \\
&= C_{X,Y}[F_1(\alpha^{-1}(x)), F_2(\beta^{-1}(y))] \\
&= C_{X,Y}(G_1(x), G_2(y)). \qquad (1.5)
\end{aligned}$$

Then,

$$C_{\alpha(X),\beta(Y)}(u, v) = C_{X,Y}(u, v), \quad \forall (u, v) \in I^2. \qquad (1.6)$$

Since X and Y are continuous, hence RanG_1 =Ran$G_2 = I = [0, 1]$. □

Theorem 1.1.5. *(Nelsen, 2006) Let X and Y be two continuous random variables with copula function $C_{X,Y}$. Let $\alpha(.)$ and $\beta(.)$ be strictly monotone on Ran(X) and Ran(Y). Then*

i) *If $\alpha(.)$ is strictly increasing and $\beta(.)$ is strictly decreasing,*

$$C_{\alpha(X),\beta(Y)}(u, v) = u - C_{X,Y}(u, 1 - v).$$

ii) *If $\alpha(.)$ is strictly decreasing and $\beta(.)$ is strictly increasing,*

$$C_{\alpha(X),\beta(Y)}(u, v) = v - C_{X,Y}(1 - u, v).$$

iii) *If $\alpha(.)$ and $\beta(.)$ are both strictly decreasing,*

$$C_{\alpha(X),\beta(Y)}(u,v) = u + v - 1 + C_{X,Y}(1-u, 1-v).$$

1.1.1 A simple proof of Sklar's theorem

Ludger R¨uschendorf (2009)

Let X be a real random varable with distribution $F(x)$ and $V \sim U(0,1)$ be uniformly distributed on $(0,1)$ and dependent of X. The modified distribution $F(x,\lambda)$ $(0 < \lambda < 1)$is dfined as :

$$F(x,\lambda) = P[X < x] + \lambda P[X = x],$$

where $P[X = x] = P[X \leq x] - P[X < x]$. We define the generalized distributional transform of X by $U := F(X,V)$. An equivalent representation of the distribution transform is.:

$$U = F(X^-) + V.(F(X) - F(X^-)) = V.F(X) + (1-V)F(X^-). \qquad (3)$$

- If $F(x)$ is continuous then $F(x,\lambda) = F(x)$, and it is well known that $U = F(X) \sim U(0,1)$.

- This property holds true for the distributional transform in general and the quantile transform is exactly the inverse of the distributional transform. Where
$$F^{-1}(u) = \inf\{x \in R : F(x) \geq u\}, \qquad u \in (0,1).$$

For the sake of completeness we give a proof of this simple but interesting result.

Lemma 1.1.6. *(Distributional transform). Let U be the distributional transform of X as defined in (3). Then*

$$U = F(X,\lambda) \sim U(0,1), \qquad \textit{and} \qquad X = F^{-1}(U), \quad \textit{a.e.}$$

Proof. For $0 < \alpha < 1$ let $q_\alpha^-(X)$ denote the lower α-quantile, that is $q_\alpha^-(X) = \sup\{: P[X \leq x]) < \alpha\}$. Then

$$F(X,V) \leq \alpha \Leftrightarrow (X,V) \in \{(x,\lambda) : P[X < x] + \lambda P[X = x] \leq \alpha\}.$$

If $\beta := P[X = q_\alpha^-(X)] > 0$ and with $q := P[X < q_\alpha^-(X)]$ this is equivalent to

$$\{X < q_\alpha^-(X)\} \cup \{X = q_\alpha^-(X)\}, q + V\beta \leq \alpha.\}$$

Thus we obtain

$$\begin{aligned} P[U \leq \alpha] &= P[F(X,\lambda) \leq \alpha] = q + \beta P\left(V \leq \frac{\alpha - q}{\beta}\right) &(1.7)\\ &= q + \beta\frac{\alpha - q}{\beta} = \alpha. &(1.8)\end{aligned}$$

If $\beta = 0$, then

$$P[F(X,\lambda) \leq \alpha] = P[X < q_\alpha^-(X)] = P[X \leq q_\alpha^-(X)] = \alpha.$$

By definition of U, we have $F(X^-) \leq U \leq F(X)$. Since for any $u \in (F(x^-), F(x))$ it hold that $x = F^{-1}(u)$. Thus we obtain that $X = F^{-1}(U), \quad a.e.$ □

The distributional transform has a lot of interesting consequences. It implies that in many respects the case of discrete or mixed type distributions does not need some extra consideration compared to the case of continuous distributions. In particular it implies a simple proof of Sklar's Theorem.

Theorem 1.1.7. *(Sklar's Theorem). Let $F \in F(F_1, ..., F_n)$ be an n-dimensional distribution function with marginals $F_1, ..., F_n$.Then there exists a copula C (i.e. an n-dimensional distribution function on I^n with uniform marginals) such that*

$$F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)), \qquad \forall x_i \in R, \qquad i = 1, 2, ..., n$$

Proof. Let $X = (X_1, ..., X_n)$ be a random vector on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with distribution function F and let V be independent of X and uniformly distributed on $(0,1)$, $V \sim U(0,1)$. Considering the distributional transforms $U_i := F_i(X_i, V), 1 \leq i \leq n$, we have by above Lemma $U_i \sim U(0,1)$, and $X_i = F_i^{-1}(U_i) a.s., 1 \leq i \leq n$. Thus defining C to be the distribution function of $U = (U_1, ..., U_n)$ we obtain

$$\begin{aligned} F(x) &= P(X \leq x) = P\left(\bigcap_{i=1}^{n}(F_i^{-1}(U_i) \leq x_i)\right) &(1.9)\\ &= P\left(\bigcap_{i=1}^{n}(U_i \leq F_i(x_i))\right) = C(F_1(x_1), ..., F_n(x_n)). &(1.10)\end{aligned}$$

Where $C(u_1, u_2, ..., u_n)$ is a copula of $F(x_1, x_2, ..., x_n)$. □

1.2 Singularity

Let X and Y be random variables with distribution functions F and G, respectively, and joint distribution function H with corresponding copula $C(.,.)$. Then in view of probability measure we have $V_H((-\infty, x] \times (-\infty, y]) = H(x, y)$, and $C_H((0, u] \times (0, v]) = C(u, v)$.

In this case for any copula $C(.,.)$, let $C(u, v) = A_C(u, v) + S_C(u, v)$, where

$$A_C(u, v) = \int_0^u \int_0^v c(t, s)dtds \quad and \quad S_C(u, v) = C(u, v) - A_C(u, v).$$

Where $c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}$.

Remark 1.2.1. 1-If $C(u, v) = A_C(u, v)$ for all $u, v \in I = [0, 1]$. Then $c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}$ and $C(u, v)$ is called absolutly continious copula.For example the copulas $\prod(u, v), \quad FGM(\theta), \; Gumble(\theta) \; and \; Gaussian(\rho)$ are absolutly continious copula.

2-If $\frac{\partial^2 C(u, v)}{\partial u \partial v} = 0$ then $C(u, v) = S(u, v)$. in this case the copula $C(u, v)$ is called singular copula.

3-Otherwisw the copula $C(u, v)$ has a absolut continious component as $A_C(u, v)$ and a singular component $S_C(u, v)$. In this case neither A_C nor S_C is not copula.

Example 1.2.2. i) The following copulas are singular: $M(u, v) = \min\{u, v\}$, because: $\frac{\partial^2 M(u, v)}{\partial u \partial v} = 0, \quad for \;\; all \;\; u \neq v$, so

$$P[U \neq V] = \int\int_{u \neq v} \frac{\partial^2 M(u, v)}{\partial u \partial v} dudv = 0$$

consequently: $P[U = V] = 1$.

ii) If $W(u, v) = \max\{u + v - 1, 0\}$, then similarly part (i) we can show that: $P[U + V = 1] = 1 - P[U + V \neq 1] = 1$.

iii) Let $C(u, v) = \sqrt{uv.M(u, v)}$ then $C(u, v) = A_C(u, v) + S_C(u, v)$. This model has a singular part on the diagonal $u = v$.

Where

$$\frac{\partial^2 C(u,v)}{\partial u \partial v} = \frac{1}{2\sqrt{v}} I_{[u<v]} + \frac{1}{2\sqrt{u}} I_{[v<u]} + h(u) I_{[v=u]}$$

and

$$h(u) = \lim_{v \to u^+} \frac{\partial C(u,v)}{\partial u} - \lim_{v \to u^-} \frac{\partial C(u,v)}{\partial u}.$$

As excersize comput $A_C(u,v)$ and $S_C(u,v)$.

1.3 Frechet-Hoeffding Bounds

The Fréchet-Hoeffding bounds are as universal bounds for copulas, i.e., for any copula $C(u,v)$ and for all $u, v \in I$,

$$W(u,v) = \max\{u + v - 1, 0\} \leq C(u,v) \leq M(u,v) = \min\{u, v\}.$$

As a consequence of Sklar's theorem, if X and Y are random variables with a joint distribution function $H(x,y)$ and margins $F(x)$ and $G(y)$ respectively, then for all $x, y \in R$,

$$H_l(x,y) = \max\{F(x) + F(y) - 1, 0\} \leq H(x,y) \leq \min\{F(x), G(y)\} = H_u(x,y).$$

Because $M(u,v)$ and $W(u,v)$ are copulas, the above bounds are joint distribution functions and are called the Fréchet-Hoeffding bounds for joint distribution functions H with margins F and G. Of interest in this section is the following question: What can we say about the random variables X and Y when their joint distribution function H is equal to one of its Fréchet-Hoeffding bounds?

Theorem 1.3.1. *Let X and Y be random variables with joint distribution function $H(x,y)$. Then $H(x,y)$ is equal to its Fréchet-Hoeffding upper bound if and only if for every $(x,y) \in R^2$, either*

$$P[X > x, Y \leq y] = 0, \quad or \quad P[X \leq x, Y > y] = 0.$$

Proof. Let $F(x)$ and $G(y)$ are mirgins of $H(x, y)$, then as application law of total probability, it is easy to write

$$F(x) = P[X \leq x] = H(x, y) + P[X \leq x, Y > y]$$

and

$$G(y) = P[Y \leq y] = H(x, y) + P[X > x, Y \leq y].$$

These imply that

$$H(x, y) = \min\{F(x), G(y)\} \Leftrightarrow P[X > x, Y \leq y] = 0 \quad or \quad P[X \leq x, Y > y] = 0.$$

□

Corollary 1.3.2. *Let X and Y be random variables with joint distribution function H. Then H is identically equal to its Fréchet-Hoeffding upper bound if and only if the support of H is a non-decreasing subset of R^2.*

Theorem 1.3.3. *Let X and Y be random variables with joint distribution function $H(x, y)$. Then $H(x, y)$ is equal to its Fréchet-Hoeffding lower bound if and only if for every $(x, y) \in R^2$, either*

$$P[X \leq x, Y \leq y] = 0 \quad or \quad P[X > x, Y > y] = 0.$$

Corollary 1.3.4. *Let X and Y be random variables with joint distribution function H. Then H is identically equal to its Fréchet-Hoeffding upper bound if and only if the support of H is a non-increasing subset of R^2.*

When X and Y are continuous, the support of $H(x, y)$ can have no horizontal or vertical line segments, and in this case it is common to say that "Y is almost surely an increasing function of X" if and only if the copula of X and Y is $M(u, v)$; and "Y is almost surely a decreasing function of X" if and only if the copula of X and Y is $W(u, v)$. If U and V are uniform $(0, 1)$ random variables whose joint distribution function is the copula $M(u, v)$, then $P[U = V] = 1$; and if the copula is $W(u, v)$, then $P[U + V = 1] = 1$. Random variables with copula $M(u, v)$ are often called comonotonic, and random variables with copula $W(u, v)$ are often called countermonotonic.

1.4 Survival copula

For a pair (X, Y) of random variables with joint distribution function $H(x, y)$, the joint survival function is given by $\bar{H}(x, y) = P[X > x, Y > y]$. The margins of $\bar{H}(x, y)$ are the functions $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(y) = 1 - G(y)$, which are the univariate survival functions $F(x)$ and $G(y)$, respectively. A natural question is the following: Is there a relationship between univariate and joint survival functions analogous to the one between univariate and joint distribution functions, as embodied in Sklar's theorem? To answer this question, suppose that the copula of X and Y is $C(u, v)$. Then we have

$$\begin{aligned} \bar{H}(x, y) &= 1 - F(x) - G(y) + H(x, y) && (1.11) \\ &= \bar{F}(x) + \bar{G}(y) - 1 + C(1 - \bar{F}(x), 1 - \bar{G}(y)), && (1.12) \end{aligned}$$

so that if we define a function if $\hat{C}(u, v)$ from I^2 to I by

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v),$$

we have

$$\bar{H}(x, y) = \hat{C}(u, v).$$

First note that, the function $\hat{C}(u, v)$ is a copula. We refer to $\hat{C}(u, v)$ as the survival copula of X and Y. Secondly, notice that $\hat{C}(u, v)$ "couples" the joint survival function to its univariate margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins. Care should be taken not to confuse the survival copula $\hat{C}(u, v)$ with the joint survival function $\bar{C}$ for two uniform $(0, 1)$ random variables whose joint distribution function is the copula C. Note that

$$\bar{C}(u, v) = P[U > u, V > v] = 1 - u - v + C(u, v) = \hat{C}(1 - u, 1 - v).$$

Let U and V be two uniform random variables on $(0, 1)$ with joint distribution function $C(u, v)$, then the survival function of C is as follows:

$$\begin{aligned} \bar{C}(u, v) = P(U > u, V > v) &= 1 - P(U \leq u) - P(V \leq v) + P(U \leq u, V \leq v) \\ &= 1 - u - v + C(u, v). && (1.13) \end{aligned}$$

So, we have

$$\bar{C}(u,1) = \bar{C}(1,v) = 0, \qquad \bar{C}(u,0) = 1-u, \qquad \bar{C}(0,v) = 1-v.$$

Let X and Y be random variables with joint distribution $F(x,y)$ and margins $F_1(x)$, $F_2(y)$ and corresponding copula $C(u,v)$, then for all $x, y \in R$, we can show that:

- $P(X \leq x, Y \leq y) = C(P(X \leq x), P(Y \leq y)) = C(F_1(x), F_2(y))$,
- $$\begin{aligned} P(X \leq x, Y > y) &= P(X \leq x) - C(P(X \leq x), 1 - P(Y > y)) \\ &= F_1(x) - C(F_1(x), 1 - \bar{F}_2(y)), \end{aligned}$$
- $$\begin{aligned} P(X > x, Y \leq y) &= P(Y \leq y) - C(1 - P(X > x), P(Y \leq y)) \\ &= F_2(y) - C(1 - \bar{F}_1(x), F_2(y)), \end{aligned}$$
- $$\begin{aligned} P(X > x, Y > y) &= \hat{C}(P(X > x), P(Y > y)) = \hat{C}(\bar{F}_1(x), \bar{F}_2(y)) \\ &= \bar{F}_1(x) + \bar{F}_2(y) - 1 + C(1 - \bar{F}_1(x), 1 - \bar{F}_2(y)). \end{aligned}$$

So if we define $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$, we have

$$\bar{F}(x,y) = \hat{C}(\bar{F}_1(x), \bar{F}_2(y)).$$

Then,

$$\hat{C}(u,v) = \bar{F}(\bar{F_1}^{-1}(u), \bar{F_2}^{-1}(v)).$$

Therefore, $\hat{C}(u,v)$ is a copula and we refer to $\hat{C}$ as the survival copula of X and Y.

Remark 1.4.1. If

$$\bar{C}(u,v) = 1 - u - v + C(u,v) = \hat{C}(1-u, 1-v)$$

then

$$C(u,v) = u + v - 1 + \hat{C}(1-u, 1-v).$$

Example 1.4.2. Let (X, Y) be a random vector with the following survival function

$$\bar{F}(x,y) = [1 + x + y + \theta xy]^{-a}; \;\; 0 \le \theta \le a+1, \; a > 0, \; x, y \ge 0.$$

Assume that $u = \bar{F}_1(x) = (1+x)^{-a}$ and $v = \bar{F}_2(y) = (1+y)^{-a}$. Then, we have

$$1 + x = u^{-1/a}, \qquad 1 + y = v^{-1/a}.$$

So,

$$x = u^{-1/a} - 1, \qquad y = v^{-1/a} - 1.$$

Therefore,

$$\begin{aligned} \hat{C}(u,v) &= \bar{F}(u^{-1/a} - 1, v^{-1/a} - 1) \\ &= [1 + u^{-1/a} - 1 + v^{-1/a} - 1 + \theta(1 - u^{-1/a})(1 - v^{-1/a})]^{-a} \\ &= [u^{-1/a} + v^{-1/a} - 1 + \theta(1 - u^{-1/a})(1 - v^{-1/a})]^{-a}; \;\; 0 < u < 1, \;\; 0 < v < 1. \end{aligned}$$

Then,

$$C(u,v) = u+v-1+\{(1-u)^{-1/a}+(1-v)^{-1/a}-1+\theta[1-(1-u)^{-1/a}][1-(1-v)^{-1/a}]\}^{-a}.$$

1.5 Symmetry

If X is a random variable and a is a real number, we say that X is symmetric about a if the distribution functions of the random variables $X - a$ and $a - X$ are the same, that is, if for any $x \in R$,

$$P[X - a \le x] = P[a - X \le x].$$

When X is continuous with distribution function F, this is equivalent to

$$F(a+x) = \bar{F}(a-x).$$

Now consider the bivariate situation. What does it mean to say that a pair (X, Y) of random variables is "symmetric" about a point (a, b)? There are a number of ways to answer this question, and each answer leads to a different type of bivariate symmetry.

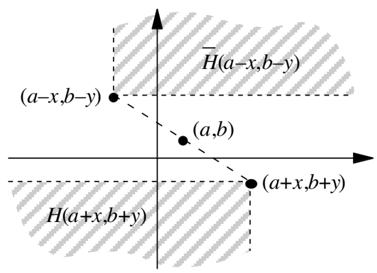
Definition 1.5.1. Let X and Y be random variables and let (a, b) be a point in R^2.
1. The random vector (X, Y) is marginally symmetric about (a, b) if X and Y are symmetric about a and b, respectively.
2.The random variable (X, Y) is radially symmetric about (a, b) if the joint distribution function of $X - a$ and $Y - b$ is the same as the joint distribution function of $a - X$ and $b - Y$.
3. The random vector (X, Y) is jointly symmetric about (a, b) if the following four pairs of random variables have a common joint distribution: $(X - a, Y - b)$, $(X - a, b - Y)$, $(a - X, Y - b)$, and $(a - X, b - Y)$.

When X and Y are continuous, we can express the condition for radial symmetry in terms of the joint distribution and survival functions of X and Y in a manner analogous to symmetry in univariate case between univariate distribution and survival functions:

Theorem 1.5.2. *Let X and Y be continuous random variables with joint distribution function H and margins F and G, respectively. Let (a, b) be a point in R^2. Then (X, Y) is radially symmetric about(a, b) if and only if*

$$H(a + x, b + y) = \bar{H}(a - x, b - y), \forall(x, y) \in R^2.$$

The term "radial" comes from the fact that the points $(a + x, b + y)$ and $(a - x, b - y)$ that appear in above formula lie on rays emanating in opposite directions from (a, b). Graphically, this Theorem states that regions such as those shaded in bellow Fig. (a) always have equal H-volume .



(a)

Example 1.5.3. The bivariate normal distribution with parameters $\mu_X, \mu_Y, \sigma_x^2, \sigma_Y^2, \rho$ is radially symmetric about the point (μ_X, μ_Y). The proof is straightforward.

Because the condition for radial symmetry in (above therem involves both the joint distribution and survival functions, it is natural to ask if copulas and survival copulas play a role in radial symmetry. The answer is provided by the next theorem.

Theorem 1.5.4. *Let X and Y be continuous random variables with joint distribution function H, marginal distribution functions F and G, respectively, and copula C. Further suppose that X and Y are symmetric about a and b, respectively. Then (X, Y) is radially symmetric about (a, b), (i.e., H satisfies in above teorem), if and only if $C = \hat{C}$, i.e., if and only if C satisfies the functional equation*

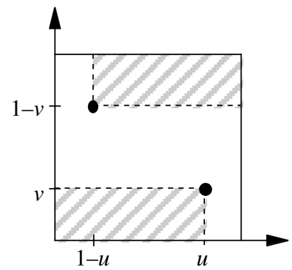
$$C(u, v) = u + v - 1 + C(1 - u, 1 - v), \forall (u, v) \in I^2. \qquad (A)$$

Proof. By applying above theorem and definition of it is easy to show that.

$$\begin{aligned} H(a + x, b + y) &= \bar{H}(a - x, b - y) \Leftrightarrow C(F(x + a), G(y + b)) \\ &= \hat{C}(\bar{F}(a - x), \bar{G}(b - y)) \Leftrightarrow C(F(x + a), G(y + b)) \\ &= \hat{C}(\bar{F}(a - x), \bar{G}(b - y)) \Leftrightarrow C(u, v) = \hat{C}(u, v). \end{aligned}$$

□

Geometrically, formula (A) states that for any $(u, v) \in I^2$, the rectangles $[0, u] \times [0, v]$and $[1-u, 1] \times [1-v, 1]$ have equal C-volume, as illustrated in Fig.(b).



(b)

Definition 1.5.5. Another form of symmetry is exchangeability—random variables X and Y are exchangeable if the vectors (X, Y) and (Y, X) are identically distributed. Hence if the joint distribution function of X and Y is H, then $H(x, y) = H(y, x)$ for all $x, y \in R^2$. Clearly exchangeable random variables must be identically distributed, i.e., have a common univariate distribution function. For identically distributed random variables, exchangeability is equivalent to the symmetry of their copula as expressed in the following theorem, whose proof is straightforward.

Theorem 1.5.6. *Let X and Y be continuous random variables with joint distribution function H, margins F and G, respectively, and copula C. Then X and Y are exchangeable if and only if $F = G$ and $C(u, v) = C(v, u)$ for all $(u, v) \in I^2$. When $C(u, v) = C(v, u)$ for all $(u, v) \in I^2$, we will say simply that C is symmetric.*

Example 1.5.7. Although identically distributed independent random variables must be exchangeable (because the copula $\prod(u, v)$ is symmetric), the converse is of course not true—identically distributed exchangeable random variables need not be independent. To show this, simply choose for the copula of X and Y any symmetric copula except $\prod(u, v)$, such as one from Example $FGM(\theta)$ or $AMH(\theta)$.

1.5.1 Order

In other words, the Fréchet-Hoeffding lower bound copula W is smaller than every copula, and the Fréchet-Hoeffding upper bound copula M is larger than every copula, i.e. $W(u,v) \leq C(u,v) \leq M(u,v)$ This point-wise partial ordering of the set of copulas is called the concordance ordering and will be important in Chapter 5 when we discuss the relationship between copulas and dependence properties for random variables (at which time the reason for the name of the ordering will become apparent).

Definition 1.5.8. If C_1 and C_2 are copulas, we say that C_1 is smaller than C_2 (or C_2 is larger than C_1), and write $C_1 \prec C_2$ (or $C_2 \prec C_1$) if

$$C_1(u,v) \leq C_2(u,v), \quad for \;\; all \;\; u,v \in I.$$

It is a partial order rather than a total order because not every pair of copulas is comparable.

Example 1.5.9. 1) The product copula $\prod(u,v)$ and the copula obtained by averaging the Fréchet-Hoeffding bounds are not comparable. If we let $C(u,v) = \frac{M(u,v)+M(u,v)}{2}$, then $C(\frac{1}{4},\frac{1}{4}) \succ \prod(\frac{1}{4},\frac{1}{4})$ and $C(\frac{1}{4},\frac{3}{4}) \prec \prod(\frac{1}{4},\frac{3}{4})$, so that neither $C \prec \prod)$ nor $C \succ \prod)$ holds.
2) However, there are families of copulas that are totally ordered. We will call a totally ordered parametric family $\{C_\theta\}$ of copulas positively ordered if $C_{\theta_1} \prec C_{\theta_2}$ whenever $\theta_1 \leq \theta_2$; and negatively ordered if $C_{\theta_1} \succ C_{\theta_2}$ whenever $\theta_1 \leq \theta_2$.
Fome more example we can check the copulas: $FGM(\theta), \quad Gumbel(\theta), \quad AMH(\theta)$.

1.6 Random variate generation

One of the primary applications of copulas is in simulation and monte Carlo studies. It is well known that, to obtain an observation x of a random variable X with distribution $F(x)$:

- Generate a variate u that is uniform on $(0, 1)$;
- Set $x = F^{-1}(u)$, where $F^{(-1)}$ is quasi-inverse of F.

There are a variety of procedures used to generate observations (x, y) of a pair or random vector (X, Y) with a joint distribution function H. In this section, we will focus on using the copula as a tool. By virtue Sklar's Theorem, we need only generate a pair (u, v) of observations of uniform $(0, 1)$ random variables (U, V) whose joint distribution function is $C(u, v)$, the copula of X and Y, and then transform those uniform variate via the algorithm such as the one in the univariate case.One procedure for generating such of a pair (u, v) of uniform $(0, 1)$ variate is the conditional distribution method. For this method , we need the conditional distribution function for V given $U = u$, which we denote:

$$C_u(v) = P[V \leq v | U = u] = \frac{\partial C(u, v)}{\partial u}.$$

Since the function $C_u(v)$, exist and is nondecreasing almost every where in v.

- Generate two independent uniform $(0, 1)$ variate u and t.
- Set $v = C_u^{-1}(t)$, where $C_u^{(-1)}$ is quasi-inverse of $C_u(.)$.
- The desired pair is (u, v), so $(x, y) = (F_1^{-1}(u), F_2^{-1}(v))$, where F_1 and F_2 are margins distribution X and Y respectively.

Example 1.6.1. Let the copula of X and Y is $C(u, v) = \frac{uv}{u+v-uv}$, then

$$C_u(v) = \frac{\partial C(u, v)}{\partial u} = \left(\frac{uv}{u + v - uv}\right)^2 \quad \textit{and} \quad C_u^{-1}(t) = \frac{u\sqrt{t}}{1 - (1 - u)\sqrt{t}}.$$

So, an algorithm to generate random variates (x, y) is:

- Generate two independent uniform $(0, 1)$ variate u and t.
- Set $\frac{u\sqrt{t}}{1-(1-u)\sqrt{t}}$.
- Set $(x, y) = (F_1^{-1}(u), F_2^{-1}(v))$, where F_1 and F_2 are margins distribution X and Y respectively.

Survival copulas can also be used in the conditional distribution function method to generate random variates from a distribution with a given survival function. If the copula C is the distribution function of a pair (U, V), then the corresponding survival copula
$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ is the distribution function of the pair $(1 - U, 1 - V)$. Also note that if U is uniform in $(0, 1)$, so is the random variable $1 - U$. Hence we have the following algorithm to generate a pair (U, V) :

- Generate two independent uniform $(0, 1)$ variate u and t.
- Set $v = \hat{C}_u^{-1}(t)$, where $\hat{C}_u^{(-1)}$ is quasi-inverse of $\hat{C}_u(.)$.
- The desired pair is (u, v), so $(x, y) = (F_1^{-1}(u), F_2^{-1}(v))$, where F_1 and F_2 are margins distribution X and Y respectively.

1.7 Multivariate Copulas

For details see the following books:
Nelsen(2006), An Introduction to copula. Chapter 2, Section 2.10. page 42.
FABRIZIO DURANTE and ,Bozen-Bolzano (2016) PRINCIPLES of COPULA THEORY , Chapter 1.

Exercises

Some problems for this chapter: Nelsen (2006). Chapter 2.

3,4,5,6,8,12,13,14,15,16,19,21,22,23,24,26,27,29,30,31